Białkowski, J., Skowroński, A. and Yamagata, K.
Osaka J. Math.
45 (2008), 159–171

CARTAN MATRICES OF SYMMETRIC ALGEBRAS HAVING GENERALIZED STANDARD STABLE TUBES

JERZY BIAŁKOWSKI, ANDRZEJ SKOWROŃSKI and KUNIO YAMAGATA

(Received June 12, 2006, revised February 26, 2007)

Abstract

We prove that the Cartan matrices of the symmetric artin algebras whose Auslander-Reiten quivers admit a generalized standard stable tube are singular and derive some consequences.

Introduction

In the paper, by an algebra is meant an artin algebra (associative, with an identity) over a commutative artin ring R. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, for $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. For an algebra A, we denote by $D\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_R(-, I)$, where I is a minimal injective cogenerator in $\operatorname{mod} R$. Further, we denote by Γ_A the Auslander-Reiten quiver of A, and by τ_A the Auslander-Reiten translation $D\operatorname{Tr}$. We will not distinguish between an indecomposable module from $\operatorname{mod} A$ and the vertex of Γ_A corresponding to it. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_{\infty}/(\tau^r)$, $r \geq 1$, is called a *stable tube* of rank r. Therefore, a stable tube of rank r in Γ_A is an infinite component consisting of τ_A-periodic indecomposable A-modules having period r. An algebra A is called *selfinjective* if the projective A-modules are injective. A distinguished class of selfinjective algebras is formed by the *symmetric algebras* for which $A \cong D(A)$ as A-A-bimodules. We also mention that, for an arbitrary algebra B, the *trivial extension* $\mathrm{T}(B) = B \ltimes D(B)$ of B by the injective cogenerator $D(B)$ is a symmetric algebra.

The Auslander-Reiten quiver is an important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A, and frequently we may recover A and $\operatorname{mod} A$ from the behaviour of distinguished components of Γ_A in the category $\operatorname{mod} A$. Following [21], a component $\mathscr{C}$ of Γ_A is called *generalized standard* if $\operatorname{rad}^{\infty}(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$. In the paper, we are concerned with the problem of describing the structure of selfinjective algebras A for which the Auslander-Reiten quiver Γ_A admits a generalized standard component, raised in [22, Problem 7].

2000 Mathematics Subject Classification. Primary 16D50, 16G70; Secondary 16D40, 16G10.

The research of the first two named authors supported by the Polish Scientific Grant KBN No. 1 P03A 018 27.

The structure of all selfinjective algebras A with Γ_A having a nonperiodic generalized standard component has been described completely in [27], [28]. On the other hand, the structure of selfinjective algebras A with Γ_A having a periodic generalized standard component is still only emerging (see [6], [7], [8], [9], [13], [15], [16], [25], [26] for some recent results in this direction).

In this paper, we are interested in the structure of symmetric algebras for which the Auslander-Reiten quiver admits a generalized standard stable tube. This is a wide class of symmetric algebras containing the trivial extensions $\mathrm{T}(B)$ of all quasitilted algebras B of canonical type over an algebraically closed field (see [1], [14], [15], [17]). We also note that an arbitrary basic finite dimensional algebra B over a field is a factor algebra of a symmetric algebra A with Γ_A having a generalized standard stable tube (see [25]).

The paper is organized as follows. In the preliminary Section 1 we present some facts on generalized standard stable tubes needed in the proof of the main result. In Section 2 we prove the main result and derive some consequences. In Section 3 we present some relevant examples illustrating our considerations.

For basic background on the representation theory of algebras applied here we refer to [2], [3], [4].

1. Stable tubes

Let A be an algebra. A module X in $\operatorname{mod} A$ is said to be a *brick* if $\operatorname{End}_A(X)$ is a division algebra. Two modules X and Y in $\operatorname{mod} A$ with $\operatorname{Hom}_A(X, Y) = 0$ and $\operatorname{Hom}_A(Y, X) = 0$ are said to be *orthogonal*. For a stable tube $\mathcal{T}$ of Γ_A the unique τ_A-orbit of $\mathcal{T}$ formed by the modules having exactly one predecessor and exactly one successor is called the *mouth* of $\mathcal{T}$.

The following characterization of generalized standard stable tubes will be critical for our considerations.

Theorem 1.1. *Let A be an algebra and $\mathcal{T}$ be a stable tube of Γ_A. The following conditions are equivalent*:
(i) *$\mathcal{T}$ is generalized standard.*
(ii) *The mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks.*

Proof. This is a part of the characterization of generalized standard stable tubes given in [21, Corollary 5.3]. In fact, in [21] only the implication (ii) $\Rightarrow$ (i) was proved in details. Because in the proof of our main result the implication (i) $\Rightarrow$ (ii) is essentially needed, we give here its detailed proof (compare the proof of [23, Proposition 3.5]).

Assume $\mathcal{T}$ is a generalized standard stable tube in Γ_A and let r be the rank of $\mathcal{T}$. Denote by $E_1, E_2, \dots, E_r$ the modules lying on the mouth of $\mathcal{T}$. We may assume that $E_i = \tau_A E_{i+1}$ for any $i \in \{1, \dots, r\}$, where $E_{r+1} = E_1$. Then, for any $i \in \{1, \dots, r\}$, we have in $\mathcal{T}$ an infinite sectional path

$$E_i = E_i[1] \to E_i[2] \to \cdots \to E_i[j] \to E_i[j+1] \to \cdots$$

called the ray of $\mathcal{T}$ starting at the mouth module E_i. Observe that every indecomposable module in $\mathcal{T}$ is of the form $E_i[j]$, for some $i \in \{1, \ldots, r\}$ and some $j \geq 1$. Moreover, we have in $\operatorname{mod} A$ almost split sequences

$$0 \to E_i[1] \to E_i[2] \to E_{i+1}[1] \to 0,$$
$$0 \to E_i[j] \to E_i[j+1] \oplus E_{i+1}[j-1] \to E_{i+1}[j] \to 0,$$

for $i \in \{1, \ldots, r\}$ and $j \geq 2$, where $E_{r+1}[j] = E_1[j]$. Then we may choose irreducible monomorphisms $u_{ij}\colon E_i[j-1] \to E_i[j]$ and irreducible epimorphisms $p_{ij}\colon E_i[j] \to E_{i+1}[j-1]$ such that $p_{i2}u_{i2} \in \operatorname{rad}^3(\operatorname{mod} A)$ and $p_{i\,j+1}u_{i\,j+1} - u_{i+1\,j}p_{ij} \in \operatorname{rad}^3(\operatorname{mod} A)$ for $i \in \{1, \ldots, r\}$ and $j \geq 2$. Observe also that, for any irreducible morphism $f\colon X \to Y$ with X and Y from $\mathcal{T}$, there are automorphisms $b\colon X \to X$ and $c\colon Y \to Y$ such that

$$f^*b + \operatorname{rad}^2(X, Y) = f + \operatorname{rad}^2(X, Y) = cf^* + \operatorname{rad}^2(X, Y),$$

where $f^*\colon X \to Y$ is the irreducible morphism of the form u_{ij} or p_{ij} chosen above. This follows from the fact that

$$\dim_{F(X)} \operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y) = 1 \quad \text{and} \quad \dim \operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y)_{F(Y)} = 1$$

where $F(X) = \operatorname{End}_A(X)/\operatorname{rad}(\operatorname{End}_A(X))$ and $F(Y) = \operatorname{End}_A(Y)/\operatorname{rad}(\operatorname{End}_A(Y))$. We note also that a morphism $f\colon X \to Y$ between indecomposable modules in $\operatorname{mod} A$ is irreducible if and only if $f \in \operatorname{rad}(X, Y) \setminus \operatorname{rad}^2(X, Y)$ (see [3, Proposition V.7.3]). Moreover, for any modules X and Y in $\operatorname{mod} A$, there exists an integer n such that $\operatorname{rad}^n(X, Y) = \operatorname{rad}^\infty(X, Y)$ (see [3, Lemma V.7.2]). Therefore, because the stable tube $\mathcal{T}$ is generalized standard, any nonisomorphism $g\colon M \to N$ with M and N from $\mathcal{T}$ is of the form $g = g_1 + \cdots + g_t$, where $g_1, \ldots, g_t$ (for some $t \geq 1$) are compositions of irreducible morphisms between indecomposable modules of the tube $\mathcal{T}$.

Let E_i and E_k, with $i, k \in \{1, \ldots, r\}$, be two modules on the mouth of $\mathcal{T}$. We may assume that $i \leq k$, and hence $E_i = \tau_A^s E_k$ for $s = k - i \geq 0$. We will show that $\operatorname{rad}(E_i, E_k) = 0$. Observe that any nontrivial path in $\mathcal{T}$ from E_i to E_k is of length $2s + 2rl$ for some $l \geq 0$. In particular, we have

$$\operatorname{rad}(E_i, E_k) = \operatorname{rad}^{2s}(E_i, E_k), \quad \text{if} \quad i \neq k,$$
$$\operatorname{rad}(E_i, E_k) = \operatorname{rad}^{2r}(E_i, E_k), \quad \text{if} \quad i = k,$$

and

$$\operatorname{rad}^{2s+2rl+1}(E_i, E_k) = \operatorname{rad}^{2s+2r(l+1)}(E_i, E_k), \quad \text{for any} \quad l \geq 0.$$

Moreover, we have $\operatorname{rad}^m(E_i, E_k) = \operatorname{rad}^\infty(E_i, E_k) = 0$ for some $m \geq 1$. Therefore, it is enough to show that $\operatorname{rad}^p(E_i, E_k) \subseteq \operatorname{rad}^{p+1}(E_i, E_k)$ for any $p \in \{1, \ldots, m-1\}$. Take $p \in \{1, \ldots, m-1\}$. We may assume that $p \geq 2s$ (for $i \neq k$) or $p \geq 2r$ (for $i = k$). Let h be a nonzero morphism from $\operatorname{rad}^p(E_i, E_k)$. Observe that $\operatorname{rad}^p(\operatorname{mod} A)$ is a left ideal of $\operatorname{mod} A$ generated by the compositions of p irreducible morphisms in $\operatorname{mod} A$. Hence $h = h_1 + \cdots + h_d$, for some $d \geq 1$, where each h_t is the composition $h_t = h_{t\,q_t} \cdots h_{t\,2} h_{t\,1}$ of a sequence of irreducible morphisms

$$E_i = X_{t\,1} \xrightarrow{h_{t\,1}} X_{t\,2} \xrightarrow{h_{t\,2}} \cdots \to X_{t\,q_t} \xrightarrow{h_{t\,q_t}} X_{t\,q_t+1} = E_k$$

with $q_t \geq p$. Then, for each $t \in \{1, \ldots, d\}$, there exists $j_t \in \{2, \ldots, q_t\}$ such that $X_{t\,j_t} = E_i[j_t]$ and $X_{t\,j_t+1} = E_{i+1}[j_t - 1]$. Then there is an automorphism a_i of $E_i = E_i[1]$ such that

$$\begin{aligned} h_t + \operatorname{rad}^{p+1}(E_i, E_k) &= h_{t\,q_t} \cdots h_{t\,j_t+1} p_{i\,j_t} u_{i\,j_t} \cdots u_{i\,2} a + \operatorname{rad}^{p+1}(E_i, E_k) \\ &= \pm h_{t\,q_t} \cdots h_{t\,j_t+1} u_{i+1\,j_t-1} \cdots p_{i\,2} u_{i\,2} a + \operatorname{rad}^{p+1}(E_i, E_k) \\ &= 0 + \operatorname{rad}^{p+1}(E_i, E_k), \end{aligned}$$

because $p_{i\,2} u_{i\,2} \in \operatorname{rad}^3(\operatorname{mod} A)$. Hence $h_t \in \operatorname{rad}^{p+1}(E_i, E_k)$. This shows that $h = h_1 + \cdots + h_d \in \operatorname{rad}^{p+1}(E_i, E_k)$. Hence, by induction on p, we conclude that $\operatorname{rad}(E_i, E_k) = \operatorname{rad}^m(E_i, E_k) = \operatorname{rad}^\infty(E_i, E_k) = 0$. Therefore, the mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks. Hence (i) implies (ii). □

We mention also that if R is an algebraically closed field K, then a stable tube $\mathcal{T}$ of Γ_A is generalized standard if and only if $\mathcal{T}$ is standard in the sense of [19] (see [24, Lemma 1.3]), that is, the full subcategory of $\operatorname{mod} A$ given by the modules of $\mathcal{T}$ is equivalent to the mesh category $K(\mathcal{T})$ of $\mathcal{T}$.

We need also the following fact.

Lemma 1.2. *Let A be a selfinjective algebra and $\mathcal{T}$ be a stable tube of Γ_A. Then the mouth of $\mathcal{T}$ contains at least one nonsimple module.*

Proof. We may assume that A is an indecomposable algebra. Let r be the rank of $\mathcal{T}$ and $E_1, \ldots, E_r$ be the modules lying on the mouth of $\mathcal{T}$ with $E_i = \tau_A E_{i+1}$ for $i \in \{1, \ldots, r\}$ and $E_{r+1} = E_1$. Assume that the modules $E_1, \ldots, E_r$ are simple. For each $i \in \{1, \ldots, r\}$, denote by P_i the projective cover of E_i in $\operatorname{mod} A$. Consider the syzygy functor Ω_A: $\underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$ on the stable category $\underline{\operatorname{mod}} A$ of $\operatorname{mod} A$, which assigns to any object M of $\underline{\operatorname{mod}} A$ the kernel $\Omega_A(M)$ of the projective cover $P(M) \to M$ of M in $\operatorname{mod} A$. Then Ω_A induces an automorphism of the stable Auslander-Reiten quiver Γ_A^s of A (see [3, Corollary X.1.10]). Hence the syzygies $\Omega_A(E_1) = \operatorname{rad} P_1, \ldots, \Omega_A(E_r) = \operatorname{rad} P_r$ of the simple modules $E_1, \ldots, E_r$ form the mouth of a stable tube of Γ_A^s with rad $P_i = \tau_A(\mathrm{rad}\, P_{i+1})$ for $i \in \{1, \dots, r\}$, and $P_{r+1} = P_1$. Applying now the shape of almost split sequences with the middle term having projective-injective direct summand (see [3, Proposition V.5.5]), we conclude that $P_i/\mathrm{soc}\, P_i \cong \mathrm{rad}\, P_{i+1}$ for all $i \in \{1, \dots, r\}$. This implies that $P_1, \dots, P_r$ are uniserial modules with the simple composition factors from the family $E_1, \dots, E_r$ of simple modules. Therefore, A is a selfinjective Nakayama algebra and $P_1, \dots, P_r$ is a complete set of pairwise nonisomorphic indecomposable projective A-module. In particular, A is of finite representation type. But this contradicts the fact that $\mathcal{T}$ is an infinite component of Γ_A. □

2. The main result

Let A be an algebra and $P_1, P_2, \dots, P_n$ be a complete set of representatives of isomorphism classes of indecomposable projective A-modules. Then $S_1 = P_1/\mathrm{rad}\, P_1$, $S_2 = P_2/\mathrm{rad}\, P_2, \dots, S_n = P_n/\mathrm{rad}\, P_n$ is a complete set of representatives of isomorphism classes of simple A-modules. For a module M in $\mathrm{mod}\, A$, denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Then $[S_1], [S_2], \dots, [S_n]$ is a $\mathbb{Z}$-basis of $K_0(A)$. Moreover, if M is a module in $\mathrm{mod}\, A$ and $[M] = m_1[S_1] + m_2[S_2] + \cdots + m_n[S_n]$ with $m_1, m_2, \dots, m_n \in \mathbb{Z}$, then $m_1, m_2, \dots, m_n$ are the multiplicities of the simple modules $S_1, S_2, \dots, S_m$ as composition factors of M. For $i, j \in \{1, \dots, n\}$, denote by c_{ij} the multiplicity of the simple module S_i as a composition factor of P_j. Then the integral $n \times n$-matrix $C_A = (c_{ij})$ is called the *Cartan matrix* of A (see [4, (1.7.9)]).

Theorem 2.1. *Let A be a symmetric algebra such that the Auslander-Reiten quiver Γ_A admits a generalized standard stable tube. Then the Cartan matrix C_A of A is singular.*

Proof. Let $\mathcal{T}$ be a generalized standard stable tube in Γ_A and r be the rank of $\mathcal{T}$. Let $E_1, \dots, E_r$ be the modules lying on the mouth of $\mathcal{T}$ with $E_i = \tau_A E_{i+1}$, for $i \in \{1, \dots, r\}$, $E_{r+1} = E_1$. Take the module $E = E_1 \oplus \cdots \oplus E_r$. Observe that $E = \tau_A E$. Since A is a symmetric algebra, we have $\tau_A E \cong \Omega_A^2 E$ (see [3, Proposition IV.3.8]). Therefore, we obtain an exact sequence

$$0 \to E \to P_1(E) \to P_0(E) \to E \to 0$$

where $P_0(E)$ is the projective cover of E and $P_1(E)$ is the injective envelope of E in $\mathrm{mod}\, A$. This leads to the equality

$$[P_1(E)] = [P_0(E)]$$

in the Grothendieck group $K_0(A)$.

Let $P_1, P_2, \dots, P_n$ be a complete set of pairwise nonisomorphic indecomposable projective A-modules, and

$$P_1(E) = m_1 P_1 \oplus \cdots \oplus m_n P_n, \quad P_0(E) = s_1 P_1 \oplus \cdots \oplus s_n P_n$$

be decompositions of $P_1(E)$ and $P_0(E)$ into direct sums of indecomposable modules, where, for a module M and $m \geq 0$, mM denotes the direct sum of m copies of M. Therefore, we obtain the equality

$$m_1[P_1] + \cdots + m_n[P_n] = s_1[P_1] + \cdots + s_n[P_n]$$

in $K_0(A)$.

Assume now that the Cartan matrix $C_A = (c_{ij})$ is nonsingular. Let $S_1, S_2, \ldots, S_n$ be the simple A-modules with $S_i = P_i/\operatorname{rad} P_i$ for any $i \in \{1, \ldots, n\}$. Observe that

$$[P_j] = c_{1j}[S_1] + c_{2j}[S_2] + \cdots + c_{nj}[S_n]$$

for any $j \in \{1, \ldots, n\}$. Because C_A is nonsingular, the columns

$$C_j = [c_{1j}, c_{2j}, \ldots, c_{nj}]^t, \quad j = 1, \ldots, n,$$

of C_A are independent in $\mathbb{Z}^n$, and consequently we obtain $m_1 = s_1, m_2 = s_2, \ldots, m_n = s_n$. Therefore, $P_1(E) \cong P_0(E)$ in $\operatorname{mod} A$. This implies that $\operatorname{soc} E \cong E/\operatorname{rad} E = \operatorname{top} E$. It follows from Lemma 1.2 that there is $i \in \{1, \ldots, r\}$ such that E_i is not simple. Then $\operatorname{rad} E_i \neq 0$ and let S be a simple direct summand of $\operatorname{top} E_i$. Because $\operatorname{soc} E \cong \operatorname{top} E$, there exists $k \in \{1, \ldots, r\}$ such that S is a direct summand of $\operatorname{soc} E_k$. Then the composed morphism

$$E_i \to \operatorname{top} E_i \to S \to \operatorname{soc} E_k \to E_k$$

is a nonzero morphism in $\operatorname{rad}(E_i, E_k)$. On the other hand, by Theorem 1.1, the generalized standardness of the stable tube $\mathcal{T}$ implies that $E_1, \ldots, E_r$ are pairwise orthogonal bricks, or equivalently $\operatorname{rad}(E, E) = 0$. Therefore, the Cartan matrix C_A is singular. □

By the remarkable theorem due to R. Brauer (see [4, Theorem 5.4.3]) the determinant of the Cartan matrix of a group algebra KG of a finite group G over a field K of characteristic $p > 0$ is a power of p. Therefore, we obtain the following fact.

Corollary 2.2. *Let K be a field of characteristic $p > 0$, G a finite group, $A = KG$ and $\mathcal{T}$ a stable tube of Γ_A. Then $\mathcal{T}$ is not generalized standard.*

We note that a group algebra KG of infinite representation type has many stable tubes (see [10], [11]).

Let B be an algebra with nonsingular Cartan matrix C_B. We may then consider the Coxeter matrix $\Phi_B = -C_B^t C_B^{-1}$ of B. We note that the Cartan matrix of any algebra of finite global dimension is nonsingular (see [2, Proposition II.3.10] or [19, p.70]).

Corollary 2.3. *Let B be an algebra with nonsingular Cartan matrix C_B and $\mathrm{T}(B) = B \ltimes D(B)$ be the trivial extension of B by $D(B)$. Assume that the Auslander-Reiten quiver $\Gamma_{\mathrm{T}(B)}$ of $\mathrm{T}(B)$ admits a generalized standard stable tube. Then 1 is an eigenvalue of the Coxeter matrix Φ_B of B.*

Proof. Since $\mathrm{T}(B)$ is a symmetric algebra, applying Theorem 2.1, we conclude that the Cartan matrix $C_{\mathrm{T}(B)}$ of $\mathrm{T}(B)$ is singular. On the other hand, it has been observed in [12, Proposition 8.2] that

$$\det C_{\mathrm{T}(B)} = (-1)^n \det C_B \det(\Phi_B - I_n),$$

where n is the rank of $K_0(A)$ and I_n is the identity matrix of degree n. Hence, we obtain $\det(\Phi_B - I_n) = 0$, and consequently 1 is an eigenvalue of Φ_B. □

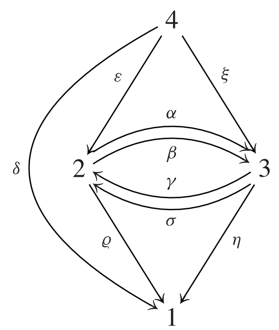
The problem of describing the selfinjective algebras with the Auslander-Reiten quiver having a generalized standard stable tube is strongly related to the problem (see [22, Problem 3]) of describing the algebras with the Auslander-Reiten quiver having a faithful generalized standard stable tube. Namely, if $\mathcal{T}$ is a generalized standard stable tube of an Auslander-Reiten quiver Γ_A and $\operatorname{ann}\mathcal{T}$ is the annihilator of $\mathcal{T}$ in A (the intersection of the annihilators of all modules in $\mathcal{T}$) then $\mathcal{T}$ is a faithful generalized standard stable tube of $\Gamma_{A/\operatorname{ann}\mathscr{C}}$. We also note that all modules in a faithful generalized standard stable tube $\mathcal{T}$ of an Auslander-Reiten quiver Γ_A have the projective dimension one and the injective dimension one (see [21, Lemma 5.9]).

3. Examples

The aim of this section is to present some examples relevant to considerations in Section 2.

We first exhibit an algebra C having singular Cartan matrix and a generalized standard stable tube in the Auslander-Reiten quiver $\Gamma_{\mathrm{T}(C)}$ of its trivial extension $\mathrm{T}(C)$.

EXAMPLE 3.1. Let K be an algebraically closed field. Consider the bound quiver algebra $C = KQ/I$ where Q is the quiver



and I is the ideal in the path algebra KQ of Q generated by the elements $\gamma\alpha$, $\alpha\gamma$, $\beta\sigma$, $\sigma\beta$, $\alpha\sigma - \beta\gamma$, $\sigma\alpha - \gamma\beta$, and $\varepsilon\alpha\sigma\varrho - \xi\gamma\beta\eta$. Then C is a generalized canonical algebra in the sense of [24, Section 2]. Indeed, let B_0 be the path algebra $K\Delta^{(0)}$ of the quiver

$$\Delta^{(0)}\colon 4 \xrightarrow{\quad\delta\quad} 1$$

and $B_1 = K\Delta^{(1)}/I^{(1)}$, where $\Delta^{(1)}$ is the quiver obtained from Q by deleting the arrow δ and $I^{(1)}$ is the ideal in $K\Delta^{(1)}$ generated by the same elements as I. Consider also the path algebra $H = K\Sigma$, where Σ is the Kronecker quiver

$$\begin{tikzcd} 2 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 3. \end{tikzcd}$$

Then the trivial extension algebra $B = \mathrm{T}(H)$ of H is the bound quiver algebra $K\Gamma/J$ where Γ is the quiver

$$\begin{tikzcd} 2 \arrow[r, bend left=40, "\alpha"] \arrow[r, bend left=20, "\beta"'] & 3 \arrow[l, bend left=20, "\gamma"'] \arrow[l, bend left=40, "\sigma"] \end{tikzcd}$$

and J is the ideal in $K\Gamma$ generated by $\gamma\alpha$, $\alpha\gamma$, $\beta\sigma$, $\sigma\beta$, $\alpha\sigma - \beta\gamma$, $\sigma\alpha - \gamma\beta$. Following notation of [24, Corollary 2.5] consider the one-point extension

$$B' = B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}$$

of B by the faithful B-module $M = B_B$. Then B' is a basic connected algebra having an indecomposable projective faithful module Q with $\operatorname{rad} Q \cong M$. Then the algebra B_1 (defined above) is the one-point coextension

$$B_1 = B'' = [Q]B' = \begin{bmatrix} B' & D(Q) \\ 0 & K \end{bmatrix},$$

the indecomposable projective B_1-module $P(4)$ at the vertex 4 is faithful and coincides with the indecomposable injective B_1-module $I(1)$ at the vertex 1. Then, by [24, Corollary 2.5], C is the generalized canonical algebra obtained from the algebras B_0 and B_1 by glueing their bound quivers at the vertices 1 and 4. Then it follows from [24, Theorem 2.1] that Γ_C admits an infinite family of pairwise orthogonal faithful (generalized) standard stable tubes. Then, by [16, Corollary 1.3], $\Gamma_{\mathrm{T}(C)}$ also admits an infinite family of pairwise orthogonal (generalized) standard stable tubes. Observe also that the

Cartan matrix of C is of the form

$$\begin{bmatrix} 1 & 4 & 4 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in the natural ordering $P(1)$, $P(2)$, $P(3)$, $P(4)$ of indecomposable projective C-modules, and is singular. We also mention that C is of infinite global dimension, because the simple modules $S(2)$ and $S(3)$ at the vertices 2 and 3 are of infinite projective dimension.

EXAMPLE 3.2. Let B be a concealed generalized canonical algebra over an algebraically closed field K, introduced in [16, Section 3]. Recall that B is an algebra of the form $\mathrm{End}_C(T)$, where C is a generalized canonical algebra, defined in [24, Section 2], and T is a tilting C-module cogenerated by the canonical family $\mathcal{T}^C$ of pairwise orthogonal faithful (generalized) standard stable tubes of Γ_C. Then, by [16, Theorem 1.1], the Auslander-Reiten quiver Γ_B of B admits a canonical family $\mathcal{T}^B$ of pairwise orthogonal faithful (generalized) standard stable tubes. Consider a selfinjective algebra A of the form $A = \hat{B}/(\varphi\nu_{\hat{B}})$, where $\hat{B}$ is the repetitive algebra of B, $\nu_{\hat{B}}$ is the Nakayama automorphism of $\hat{B}$ and φ is a positive automorphism of $\hat{B}$. We note that the induced Galois covering $\hat{B} \to \hat{B}/(\varphi\nu_{\hat{B}}) = A$ is a positive Galois covering in the sense of [29]. It has been proved in [16, Theorem 1.2] that the Auslander-Reiten quiver Γ_A of A admits an infinite family of pairwise orthogonal (generalized) standard stable tubes. We also note that A is symmetric if and only if $A \cong \mathrm{T}(B)$ (see [18, Theorem 2]). Therefore, if φ is a strictly positive automorphism of $\hat{B}$, then A is a nonsymmetric selfinjective algebra with Γ_A having an infinite family of (generalized) standard stable tubes. We refer to [18], [26] and [29] for more details on selfinjective orbit algebras of repetitive algebras.

Assume now that B is of finite global dimension, and hence the Coxeter matrix Φ_B is defined. We note that this is the case if the generalized canonical algebra C is of finite global dimension. We know from [16, Section 4] that Γ_B admits a faithful generalized standard stable tube $\mathcal{T}$ of rank one. Then $\mathrm{Hom}_B(\mathcal{T}, B_B) = 0$ and $\mathrm{pd}_B X \leq 1$ for any module X in $\mathcal{T}$ (see [21, Lemma 5.9]). Take a module X in $\mathcal{T}$. Since B is a basic algebra, $[X]$ is the dimension vector $\underline{\dim}\, X$ of X, under the canonical identification $K_0(A) = \mathbb{Z}^n$. Applying now [2, Corollary IV.2.9], we conclude that $\underline{\dim}\, X = \underline{\dim}\, \tau_A X = \Phi_B\, \underline{\dim}\, X$, and consequently $\underline{\dim}\, X$ is an eigenvector of Φ_B with eigenvalue 1.

For each $m \geq 2$, consider the selfinjective orbit algebra $\Lambda_B^{(m)} = \hat{B}/(\nu_{\hat{B}}^m)$. It follows from [12, Proposition 8.2] that the determinant of $\Lambda_B^{(m)}$ is of the form

$$(-1)^{mn}(\det C_B)^m \prod_{r=1}^{m} \det(\Phi_B - \varepsilon_r I_n)$$

where $\varepsilon_1, \ldots, \varepsilon_m$ are distinct m-th roots of unity, and I_n is the identity matrix of degree n.

Therefore, for any concealed generalized canonical algebra B of finite global dimension, the algebras $\Lambda_B^{(m)}$, $m \geq 2$, are nonsymmetric selfinjective algebras with singular Cartan matrices and the Auslander-Reiten quivers having generalized standard stable tubes.

In the final example we show that there exist nonsymmetric selfinjective algebras with nonsingular Cartan matrices for which the Auslander-Reiten quiver admits a generalized standard stable tube. This will show that the symmetricity assumption is necessary for the validity of Theorem 2.1.

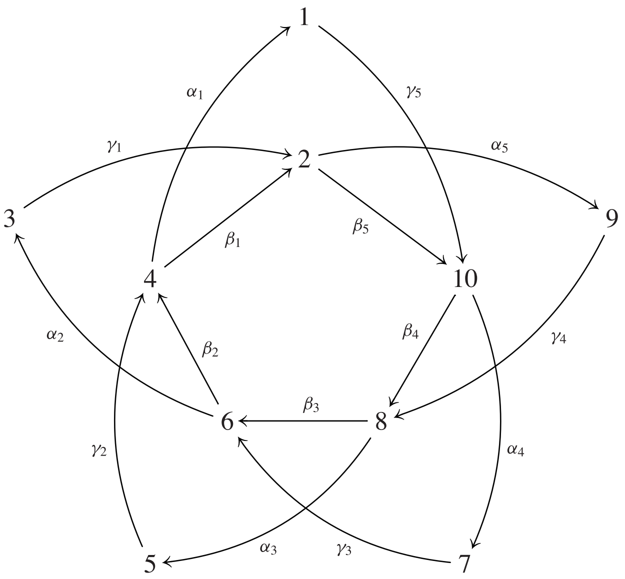
EXAMPLE 3.3. Let K be an algebraically closed field. Consider the bound quiver algebra $B = KQ/I$ where Q is the quiver

$$\begin{tikzcd}
7 \arrow[dr, "\gamma_3"] & & 8 \arrow[dl, "\beta_3"'] \arrow[dr, "\alpha_3"] & \\
& 6 \arrow[dl, "\alpha_2"] \arrow[dr, "\beta_2"] & & 5 \arrow[dl, "\gamma_2"] \\
3 \arrow[dr, "\gamma_1"] & & 4 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_1"] & \\
& 2 & & 1
\end{tikzcd}$$

and I is the ideal in the path algebra KQ of Q generated by the elements $\gamma_3\beta_2\alpha_1$, $\alpha_2\gamma_1 - \beta_2\beta_1$, $\beta_3\beta_2 - \alpha_3\gamma_2$. Then B is the exceptional (in the sense of [20, (3.2)]) tubular algebra B_4 of tubular type (3, 3, 3) presented in [7, Theorem 2.2]. It follows from [7, Section 3] that the Nakayama automorphism $\nu_{\hat{B}}$ of $\hat{B}$ admits a 4-root φ. For each $i \geq 1$, consider the selfinjective orbit algebra $\Omega_B^{(i)} = \hat{B}/(\varphi^i)$, and note that $\Omega_B^{(4)} = \hat{B}/(\nu_{\hat{B}}) = \mathrm{T}(B)$. It follows from the theory of selfinjective algebras of tubular type (see [6], [17], [20]) that, for $i \geq 4$, the Auslander-Reiten quiver of $\Omega_B^{(i)}$ admits an infinite family of generalized standard stable tubes. On the other hand, from the description of the determinants of Cartan matrices of selfinjective algebras of tubular type given in [5, Theorem], we know that, in the considered case, we have

$$\det C_{\Omega_B^{(i)}} = \begin{cases} 6 & \text{if} \quad i \equiv \pm 1 \pmod 6 \\ 12 & \text{if} \quad i \equiv \pm 2 \pmod{12} \\ 0 & \text{in other case} \end{cases}.$$

In particular, taking $A = \Omega_B^{(5)}$, we conclude that A is a nonsymmetric selfinjective algebra with $\det C_A = 6$ and the Auslander-Reiten quiver Γ_A having an infinite family of generalized standard stable tubes. In fact, A is the bound quiver algebra $K\Delta/J$ where Δ is the quiver



and J is the ideal in $K\Delta$ generated by the elements $\gamma_{i+2}\beta_{i+1}\alpha_i$, $\alpha_{i+1}\gamma_i - \beta_{i+1}\beta_i$, for $i \in \{1, \dots, 5\}$ with $\alpha_6 = \alpha_1$, $\beta_6 = \beta_1$, $\gamma_7 = \gamma_2$. Moreover, the Cartan matrix C_A is of the form (in the canonical numbering of indecomposable projective A-modules)

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}.$$

References

[1] I. Assem, J. Nehring and A. Skowroński: *Domestic trivial extensions of simply connected algebras*, Tsukuba J. Math. **13** (1989), 31–72.

[2] I. Assem, D. Simson and A. Skowroński: Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Mathematical Society Student Texts **65**, Cambridge Univ. Press, Cambridge, 2006.
[3] M. Auslander, I. Reiten and S.O. Smalø: Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics **36**, Cambridge Univ. Press, Cambridge, 1995
[4] D.J. Benson: Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras, Cambridge Studies in Advanced Mathematics **30**, Cambridge Univ. Press, Cambridge, 1991.
[5] J. Białkowski: *Cartan matrices of selfinjective algebras of tubular type*, Cent. Eur. J. Math. **2** (2004), 123–142.
[6] J. Białkowski and A. Skowroński: *Selfinjective algebras of tubular type*, Colloq. Math. **94** (2002), 175–194.
[7] J. Białkowski and A. Skowroński: *On tame weakly symmetric algebras having only periodic modules*, Arch. Math. (Basel) **81** (2003), 142–154.
[8] R. Bocian and A. Skowroński: *Symmetric special biserial algebras of Euclidean type*, Colloq. Math. **96** (2003), 121–148.
[9] R. Bocian and A. Skowroński: *Weakly symmetric algebras of Euclidean type*, J. Reine Angew. Math. **580** (2005), 157–199.
[10] K. Erdmann: Blocks of Tame Representation Type and Related Algebras, Lecture Notes in Math. **1428**, Springer-Verlag, Berlin, 1990.
[11] K. Erdmann: *On tubes for blocks of wild type*, Colloq. Math. **82** (1999), 261–270.
[12] K. Erdmann, O. Kerner and A. Skowroński: *Self-injective algebras of wild tilted type*, J. Pure Appl. Algebra **149** (2000), 127–176.
[13] O. Kerner, A. Skowroński and K. Yamagata: *Invariance of selfinjective algebras of quasitilted type under stable equivalences*, Manuscripta Math. **119** (2006), 359–381.
[14] H. Lenzing and A. Skowroński: *Quasitilted algebras of canonical type*, Colloq. Math. **71** (1996), 161–181.
[15] H. Lenzing and A. Skowroński: *Selfinjective algebras of wild canonical type*, Colloq. Math. **96** (2003), 245–275.
[16] P. Malicki and A. Skowroński: *Concealed generalized canonical algebras and standard stable tubes*, J. Math. Soc. Japan **59** (2007), 521–539.
[17] J. Nehring and A. Skowroński: *Polynomial growth trivial extensions of simply connected algebras*, Fund. Math. **132** (1989), 117–134.
[18] Y. Ohnuki, K. Takeda and K. Yamagata: *Automorphisms of repetitive algebras*, J. Algebra **232** (2000), 708–724.
[19] C.M. Ringel: Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math. **1099**, Springer-Verlag, Berlin, 1984.
[20] A. Skowroński: *Selfinjective algebras of polynomial growth*, Math. Ann. **285** (1989), 177–199.
[21] A. Skowroński: *Generalized standard Auslander-Reiten components*, J. Math. Soc. Japan **46** (1994), 517–543.
[22] A. Skowroński: *Cycles in module categories*; in Finite-Dimensional Algebras and Related Topics (Ottawa, ON, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **424**, Kluwer Acad. Publ., Dordrecht, 1994, 309–345.
[23] A. Skowroński: *On the composition factors of periodic modules*, J. London Math. Soc. **49** (1994), 477–492.
[24] A. Skowroński: *Generalized canonical algebras and standard stable tubes*, Colloq. Math. **90** (2001), 77–93.
[25] A. Skowroński: *A construction of complex syzygy periodic modules over symmetric algebras*, Colloq. Math. **103** (2005), 61–69.
[26] A. Skowroński: *Selfinjective algebras*: *finite and tame type*, Contemp. Math. **406** (2006), 169–238.
[27] A. Skowroński and K. Yamagata: *Galois coverings of selfinjective algebras by repetitive algebras*, Trans. Amer. Math. Soc. **351** (1999), 715–734.
[28] A. Skowroński and K. Yamagata: *On selfinjective artin algebras having nonperiodic generalized standard Auslander-Reiten components*, Colloq. Math. **96** (2003), 235–244.

[29] A. Skowroński and K. Yamagata: *Positive Galois coverings of selfinjective algebras*, Adv. Math. **194** (2005), 398–436.

Jerzy Białkowski
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87–100 Toruń
Poland
e-mail: jb@mat.uni.torun.pl

Andrzej Skowroński
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87–100 Toruń
Poland
e-mail: skowron@mat.uni.torun.pl

Kunio Yamagata
Department of Mathematics
Tokyo University of Agriculture and Technology
Nakacho 2–24–16, Koganei
Tokyo 184–8588
Japan
e-mail: yamagata@cc.tuat.ac.jp